ON THE UNIFORMIZATION OF ALGEBRAIC FUNCTIONS.

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In his investigations on the uniformization of algebraic functions by means of automorphic functions with domain of definition T, *Koebe** has solved two central problems.

I. THE DOMAIN T, BOUNDED BY A CIRCLE.

THEOREM I. *Let w be an algebraic function of z of deficiency $p > 1$. Then there exist two automorphic functions, $\varphi(t)$, $\psi(t)$, whose domain of definition T is the interior of the unit circle, $|t| < 1$, and such that the algebraic configuration $\mathfrak{A}$ whose points are (w, z) is represented by the pair of equations:*

$$z = \varphi(t), \qquad w = \psi(t).$$

To each point t_0 within T corresponds a single point (w_0, z_0) of $\mathfrak{A}$. Conversely, to each point (w_0, z_0) of $\mathfrak{A}$ correspond points t_0, t_0', $\cdots$, interior to T and having as their points of condensation the boundary of T.

Finally, to a certain neighborhood of (w_0, z_0) corresponds a region of T including t_0 in its interior and such that the relation between the points of the two regions in question is one-to-one.

II. THE DOMAIN T, BOUNDED BY THE POINTS OF A DISCRETE SET.

A set of points in the t-plane shall be said to be *discrete* when it satisfies the following conditions. Let $t = a$ be any finite point of the set, and let a circle of arbitrarily small radius be drawn about a as center. Then it shall be possible to draw a simple closed curve inside this circle, which encloses the point a and does not go through either a point of the set or a cluster point of the set.

It is clear that the points of the plane that remain after the points of a discrete set and their limiting points have been removed constitute a single continuum; and, furthermore, that this latter region extends into every neighborhood of each point of the plane.

In the case of the second theorem T is a region whose boundary consists of a discrete set of points.

* Koebe, Mathematische Annalen, vol. 67 (1909), p. 145 and vol. 69 (1910), p. 1. These papers were preceded by a series of notes in the Göttinger Nachrichten beginning in 1907.

THEOREM II. *Let w be an algebraic function of z of deficiency $p > 0$. Then there exist two automorphic functions,* $\varphi(t)$, $\psi(t)$, whose domain of definition T is bounded by the points of a discrete set, and which are such that the algebraic configuration $\mathfrak{A}$ whose points are (w, z) is represented by the pair of equations*

$$z = \varphi(t), \qquad w = \psi(t)$$

in the same sense as in Theorem I.

Koebe has collected his earlier results and given a systematic treatment of Theorem I and related problems in the first of the *Annalen* papers above cited. To cull from the eighty pages of this article that which is essential in the methods is a task of some labor, and furthermore the proofs admit of simplification. In the second edition of my *Funktionentheorie*, vol. 1, I have given what seems to me to be a simple and complete treatment of this problem. The treatment is also typical for the other problems of *Koebe's* first article. It is the purpose of the present paper to help the reader to do the same thing for Theorem II and the related problems.

§ 1. The Surfaces Φ_n and $\Phi = \lim\limits_{n=\infty} \Phi_n$.

Consider an algebraic Riemann's surface F,—i. e., a Riemann's surface corresponding to an algebraic function,—of deficiency $p > 0$. It is possible to draw p loop cuts in it† and still not have it fall apart. This is, however, the maximum number of such cuts that can be drawn in F. The surface thus cut, F_1, is called a surface of *planar character* (*schlichtartig*), and this term will be applied to any surface of two sides which is cut in two by every loop cut. The loop cuts may be taken as analytic curves lying in the finite region.‡

Suppose we have an unlimited number of duplicates of F_1. Let a surface Φ_2 be constructed by extending F_1 by $2p$ duplicates, one being joined to F_1 along the opposite side of each bank of each of the p cuts. The resulting surface is also of planar character, and like F_1 it has a finite number of

* By an *automorphic function* shall be understood in this paper a *single-valued* automorphic function.

† When a simultaneous system of loop cuts is drawn in a surface, it is understood that they are not to intersect one another, and that each loop cut is a simple curve on the surface.

‡ A loop cut may be taken, first, as a polygon whose sides are each parallel to one of the coordinate axes. The vertices can now be smoothed off by easement curves, so that the new loop cut has continuous curvature. The coordinates of the latter cut can be represented parametrically by two functions, each of which can be developed into a Fourier's series differentiable term by term, the derivative series converging uniformly. A suitable number of terms from each of these developments being chosen and their sum being set equal to the proper coordinate, the new curve thus defined is a loop cut lying uniformly arbitrarily near to the original cut and being simple and analytic throughout.

leaves and branch points, and is bounded by a finite number of closed analytic curves. Moreover, F_1, inclusive of its boundary, lies within Φ_2.

Next, construct a surface Φ_3 out of Φ_2 in a precisely similar manner by extending Φ_2 across each of its boundaries by means of further duplicates of F_1. By repeating the process, an unlimited sequence of surfaces

$$\Phi_1 = F_1, \quad \Phi_2, \quad \Phi_3, \quad \cdots$$

is obtained, each with the above mentioned characteristics of Φ_1 and Φ_2. The limiting surface shall be denoted by Φ.

§ 2. The Map of Φ_n on a Plane Region.

We proceed next to the consideration of the following theorem.

The region Φ_n can be mapped by a function

$$t = F_n(z)$$

in a one-to-one manner and continuously, and in general conformally, on a region T_n consisting of the extended single-leaved t-plane with the exception of slits. The latter consist of segments of right lines, finite in number and parallel to the axis of reals, and they correspond to the boundary curves of Φ_n. Furthermore, an ordinary interior point O of Φ_1 having been chosen at pleasure and the origin $z = 0$ having been transformed to it, the neighborhood of O in its sheet of Φ_1 is mapped on the neighborhood of the point $t = \infty$ as follows:

$$t = F_n(z) = \frac{1}{z} + \omega_n(z),$$

where $\omega_n(z)$ remains finite in the neighborhood of O.

Finally, one of these segments shall pass through the point $t = 0$.*

To prove this theorem, we begin by constructing a function η_0 which shall be single-valued and continuous and in general harmonic on the surface Φ_n, except at the point O, where

$$(1) \qquad \eta_0 = -\frac{\sin\theta}{r} + \mu(x, y),$$

μ remaining finite at O. On the boundary C of Φ_n, η_0 shall vanish:†

$$\eta_0 \big|_C = 0.$$

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* A proof of this theorem is sketched by Koebe in the second of the *Annalen* papers above cited, § 13. The problem has also been treated by Cecioni, Rend. Circ. Mat. Palermo, vol. 25 (1908), p. 1; Hilbert, Göttinger Nachrichten, 1909, p. 314, and Courant, Göttingen Dissertation, Math. Annalen, vol. 72 (1912), p. 517.

† The existence of such a function is established by the method of successive approximations as applied by Schwarz and Neumann. For the details of the proof cf. Osgood, Lehrbuch der Funktionentheorie, vol. 1, 2d ed., 1912, Ch. 14, § 9, in particular, footnote, p. 717.

Let ξ_0 be the negative of the function conjugate to η_0. Then, in the neighborhood of O,

$$\xi_0 = \frac{\cos\theta}{r} + \lambda(x, y), \tag{2}$$

λ remaining finite there. In general ξ_0 will be multiple-valued on Φ_n. We proceed to replace η_0 by such a function η_n that its conjugate will be single-valued.

Denote the curves that form the boundary of Φ_n by $C_1, \cdots, C_N$. Let $v_1, \cdots, v_N$ be functions single-valued and continuous in Φ_n, and harmonic in the ordinary points of Φ_n, and let

$$v_j\big|_{C_j} = 1; \qquad v_j\big|_{C_k} = 0, \qquad j \neq k.$$

Let $u_1, \cdots, u_N$ be the negatives of the functions which are conjugate respectively to $v_1, \cdots, v_N$. Denote the moduli of periodicity of $u_1, \cdots, u_N$ along (not across) the curve C_j, when that curve is described in the positive sense, by

$$\omega_1^{(j)}, \quad \cdots, \quad \omega_N^{(j)}. \tag{3}$$

Let $c_1, \cdots, c_N$ be N arbitrary constants, and form the functions

$$u = c_1 u_1 + \cdots + c_N u_N,$$

$$v = c_1 v_1 + \cdots + c_N v_N.$$

The modulus of periodicity of u along the curve C_j is

$$c_1 \omega_1^{(j)} + c_2 \omega_2^{(j)} + \cdots + c_N \omega_N^{(j)}. \tag{4}$$

Some of these quantities may vanish, but not all of them, except when all the c's are 0. For, if they did, u would be single-valued. Consider the curve

$$\Gamma: \qquad v = c \neq c_j, \qquad j = 1, 2, \cdots N.$$

Here, Γ is a regular closed curve* on Φ_n, and since Φ_n is of planar character, Γ divides it into two or more pieces. Hence a connected region of Φ_n has as one of its boundaries a part or the whole of Γ, and in this region $v - c$ is always positive, or else always negative. From this it follows that $\partial v/\partial \nu$, where ν denotes the inner normal, does not change sign along the boundary in question, and it cannot vanish along any arc of this boundary, however short.† But the change in the conjugate function $-u$ is given

* The discussion of the isothermals of the Green's function, Funktionentheorie, Ch. 13, § 7, applies to the present locus.

† Funktionentheorie, p. 665.

precisely by the integral

$$-\int \frac{\partial v}{\partial \nu} ds$$

taken over the boundary in question, and here is a contradiction.

As a consequence of the foregoing we have the relation:

$$\begin{vmatrix} \omega_1^{(1)} & \cdots & \omega_N^{(1)} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \omega_1^{(N)} & \cdots & \omega_N^{(N)} \end{vmatrix} \neq 0.$$

Returning now to the functions η_0, ξ_0, let us form the functions

$$\xi_n = \xi_0 + c_1 u_1 + \cdots + c_N u_N + a,$$

$$\eta_n = \eta_0 + c_1 v_1 + \cdots + c_N v_N + b.$$

We perceive that it is always possible so to determine the c's that all the moduli of periodicity of ξ_n shall vanish, and thus ξ_n will be a single-valued function on the surface Φ_n. Let this be done, and furthermore let a, b be so chosen that ξ_n, η_n both vanish in a given boundary point of Φ_n. We note that

$$\eta_n\big|_{C_j} = c_j + b = c_j', \qquad j = 1, 2, \cdots, N.$$

Finally, the function

$$t = \xi_n + i\eta_n = F_n(z) \tag{5}$$

yields the desired map. For, consider the curve

$$K: \qquad \eta_n = c \neq c_j', \qquad j = 1, 2, \cdots, N.$$

This curve, as in the case of Γ above, is seen to be a regular closed curve on the surface, and it passes through O, having there a single branch. Since Φ_n is of planar character, it is cut in two by K. Moreover, K is simple on the surface. For otherwise Φ_n would be cut into more than two pieces by K, and one of these pieces, then, would not abut on O. In this piece, $\eta_n - c$ would be always positive, or else always negative, and thus we are led to a similar situation and contradiction to that on which was based the proof above, that not all the quantities (4) vanish.

Since the value of ξ_n along K is given by the integral

$$\xi_n = \int \frac{\partial \eta_n}{\partial \nu} ds + \text{const.}$$

extended along a variable arc of K, the proper normal ν being chosen, and since $\partial\eta_n/\partial\nu$ does not change sign along K or vanish along any arc of K, it follows, with reference to the relations (1) and (2), that the points of K

are transformed by (5) in a one-to-one manner and continuously into the points of the right line of the $t = \xi + \eta i$-plane:

$$\eta = c.$$

Thus a one-to-one relation is established between all the points of the surface Φ_n for which $\eta_n \neq c_j'$, $j = 1, 2, \cdots, N$, and the points of the extended t-plane exclusive of the N parallels to the axis of reals, $\eta = c_j'$.

To deal with the excepted points let P be an interior point of Φ_n not a branch point or a point ∞, in which $\eta_n = c_j'$. The function (5) maps the neighborhood of P either on a single-leaved region of the t-plane or on the neighborhood of a branch point. From the foregoing result the latter alternative is seen to be impossible. Hence the points of the curve $\eta_n = c_j'$ which lie in the neighborhood of P go over in a one-to-one manner into the points of a segment of the line $\eta = c_j'$. In particular, then, it follows that $\partial\eta_n/\partial x$ and $\partial\eta_n/\partial y$ never vanish simultaneously in a point, P,—or in fact in any finite point of Φ_n. Hence the interior points of Φ_n which lie on the curves $\eta_n = c_j'$ go over in a one-to-one manner into the points of segments of the lines $\eta = c_j'$.

This completes the proof for interior points. It is easily seen that the points of the boundary curves C_j go over in a one-to-two manner into segments of the lines $\eta = c_j'$.

§ 3. The Map of Φ on a Single-Leaved Region.

Let

$$f_n(z) = \frac{1}{F_n(z)}, \qquad z \neq 0; \qquad f_n(0) = 0,$$

the exceptional point $z = 0$ being merely the point O of a single leaf. Then the function

$$t = f_n(z)$$

maps Φ_n on a single-leaved region R_n of the t-plane in a one-to-one manner and continuously, and in general conformally, each interior point of Φ_n going over into a finite point t. Moreover,

$$f_n(0) = 0, \qquad f_n'(0) = 1.$$

The proof of Theorem II turns on the following theorem.

Let Φ_k be chosen arbitrarily from the regions $\Phi_1, \Phi_2, \cdots$. From the set of functions $f_1(z), f_2(z), \cdots$ a set

$$f_{n_1}(z), \qquad f_{n_2}(z), \qquad \cdots \qquad n_i < n_{i+1},$$

can then be selected which converges uniformly in Φ_k. The indices n_i are

independent of k; and it is, moreover, understood that those functions, if any such exist, for which $n_i < k$ are to be omitted.

The limiting function,

$$f(z) = \lim_{i=\infty} f_{n_i}(z),$$

is uniquely defined at each point of Φ, and by means of it,

$$t = f(z),$$

Φ is mapped on a single-leaved region T of the t-plane as required in Theorem II.

To prove this theorem we begin by showing, in the next paragraph, that the functions $f_n(z)$, $k \leqq n$, remain finite in Φ_k.

§ 4. A Lemma.

Consider the functions

$$(1) \qquad t = f(z)$$

which map a given circle $|z| < \rho$ on single-leaved regions S not containing the point $t = \infty$ in their interior (though it may lie on the boundary), and which, furthermore, are such that

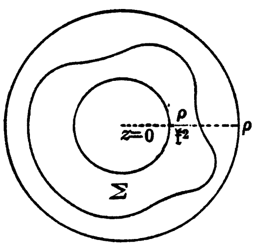
$$f(0) = 0, \qquad f'(0) = 1.$$

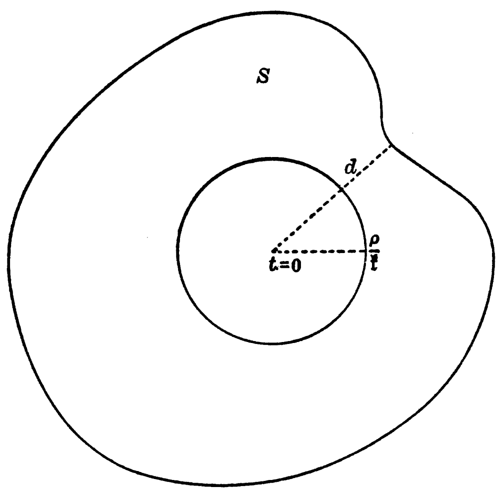
These functions remain finite in the circle $|z| \leqq \rho/\mathfrak{k}^2$, where $\mathfrak{k}$ denotes Koebe's constant; in fact*

$$|f(z)| \leqq \frac{\rho}{\mathfrak{k}}, \qquad \textit{where} \qquad |z| \leqq \frac{\rho}{\mathfrak{k}^2}.$$

Moreover,

$$|f'(z)| \leqq 4\mathfrak{k}, \qquad \textit{where} \qquad |z| \leqq \frac{\rho}{2\mathfrak{k}^2}.$$





* Funktionentheorie, p. 727.

Let d be the distance from the point $t = 0$ to the boundary of S. Then*

$$d \geqq \frac{\rho}{\mathfrak{k}}.$$

Hence the interior of the fixed circle

$$(2) \qquad |t| \leqq \frac{\rho}{\mathfrak{k}}$$

is mapped by each of the functions (1) on a region Σ of the z-plane.

Let δ be the distance from $z = 0$ to the boundary of Σ. Then, by a second application of the above theorem

$$\delta \geqq \frac{\rho}{\mathfrak{k}^2},$$

and hence the fixed circle

$$(3) \qquad |z| \leqq \frac{\rho}{\mathfrak{k}^2}$$

is mapped by each of the functions (1) on a regon of the t-plane lying in the fixed circle (2). Thus the first part of the theorem is established.

To prove the second part, represent $f'(z)$ in the circle

$$(4) \qquad |z| \leqq \frac{\rho}{2\mathfrak{k}^2}$$

by the integral:

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2},$$

extended over the circle C: $|z| = \rho/\mathfrak{k}^2$. Here,

$$|f(\zeta)| \leqq \frac{\rho}{\mathfrak{k}}, \qquad |\zeta - z| \geqq \frac{\rho}{2\mathfrak{k}^2}, \qquad |d\zeta| = \frac{\rho}{\mathfrak{k}^2}\,d\varphi,$$

and from these relations the proof follows at once.

From the results just obtained we can now deduce the following theorem, which includes the foregoing as a special case.

Let Σ be a region spread out over the extended z-plane, having a finite number of leaves and branch points, and bounded by a finite number of regular curves, none of which go through a branch point or the point ∞. Let Σ' be a second region containing Σ in its interior, but having no branch point on the boundary of Σ, and let Σ, Σ' be such that Σ' can be mapped on a single-leaved plane region.

Consider the functions

$$t = f(z)$$

* Funktionentheorie, p. 726, § 12. This theorem, which is due in substance to Koebe, is fundamental both in the investigations referred to at the beginning of this paper and in those in which we are now engaged.

which map Σ' on single-leaved regions of the t-plane not including the point $t = \infty$ in their interior, and which, furthermore, are such that, at an ordinary interior point O, $z = 0$, of Σ

$$f(0) = 0, \qquad f'(0) = 1.$$

These functions $f(z)$ remain finite in Σ.

Let the neighborhoods of the points $z = \infty$ and of the branch points be removed from Σ, the new region $\bar{\Sigma}$, which is finite and without branch points, having as its additional boundaries circles. It is evidently sufficient to show that, *in the region $\bar{\Sigma}$, the functions $f(z)$ remain finite.*

It is possible to divide the plane up into a net work of squares, of length h on a side, and such that the various leaves of $\bar{\Sigma}$ are divided by the projection of the boundaries of these squares into a finite number of squares or pieces of squares in the following manner. A circle of radius

$$\rho = 2\sqrt{2}h \cdot 2\mathfrak{k}^2 = 4\sqrt{2}\mathfrak{k}^2 h$$

with its center within or on the boundary of a square which includes interior points of $\bar{\Sigma}$ shall not reach out in its leaf to a branch point or boundary point of Σ'. In a square which contains O in its interior or on its boundary the functions $f(z)$ are seen from the theorem at the beginning of the paragraph to remain finite, and moreover $|f'(z)| \leqq 4\mathfrak{k}$ there.

Let s be any one of the squares in which $f(z)$ and $f'(z)$ remain finite; s_1, an adjacent square, and z_1 a point on the boundary of each. These squares shall be considered as including their boundaries. Then $f(z)$ and $f'(z)$ remain finite in s_1. For, $f'(z_1) \neq 0$, and the function

$$\bar{f}(z) = \frac{f(z) - f(z_1)}{f'(z_1)}$$

satisfies the conditions of the theorem proven at the beginning of the paragraph, the point z_1 corresponding to O.

Since there are only a finite number of squares to be considered, the finiteness of $f(z)$ and $f'(z)$ in their totality is herewith established, and thus the theorem is proven. Incidentally we have obtained the further result: *In the region $\bar{\Sigma}$ the functions $f'(z)$ also remain finite.*

We are now in a position to take the first step in the proof of the theorem of § 3, indicated at the close of that paragraph. Let Σ be taken as the domain Φ_k, and Σ' as a domain Φ_m, $m > k$. The functions $f_n(z)$, $n \geqq m$, are then included in the class of functions $f(z)$ of the theorem of the present paragraph, and hence we see that the functions $f_n(z)$, $n > k$, remain finite in Φ_k.

The next step is to show that from these functions a set $f_{n_i}(z)$ can be selected which converge uniformly in Φ_k.

§ 5. A Theorem in Uniform Convergence.

We will begin by stating a theorem for a real function of a real variable, which contains that which is essential in the theorem that follows later in the paragraph.

Let $f_n(x)$ be a real function of the positive integer n and the real variable x in the finite closed interval $a \leqq x \leqq b$; and let

(a) $f_n(x)$, regarded as a function of x and n, be finite:

$$|f_n(x)| < M, \qquad a \leqq x \leqq b, \qquad n = 1, 2, 3, \cdots,$$

M being a positive constant;

(b) let the difference quotient also remain finite

$$\left|\frac{f_n(x') - f_n(x'')}{x' - x''}\right| < M',$$

where x' and x'' are any two distinct points of the above interval, and n is arbitrary; M' being a positive constant.

Then it is possible to choose from the functions $f_n(x)$ a set

$$f_{n_1}(x), \quad f_{n_2}(x), \quad \cdots,$$

*which converges uniformly in the above interval (a, b).**

From (b) it follows that $f_n(x)$ is a continuous function of x in the closed interval (a, b). Moreover, Condition (b) will always be fulfilled when $f_n(x)$ possesses a derivative which, regarded as a function of x and n, remains finite.

To prove the theorem, let $a_1, a_2, \cdots$ be a set of distinct points of (a, b) everywhere dense in this interval. For convenience, take them as the points of division when the interval is divided into 2^k equal intervals, $k = 1, 2, 3, \cdots$.

From the set of functions $f_n(x)$ it is possible to choose a set which we will denote by

(1) $$f_{1,1}(x), \quad f_{1,2}(x), \quad \cdots,$$

and which converge for the value of the argument $x = a_1$.

From the set (1) of functions we can now select a set

(2) $$f_{2,1}(x), \quad f_{2,2}(x), \quad \cdots$$

* The theorem and its proof are closely related to the investigations of de la Vallée-Poussin on differential equations, 1892; Mém. Acad. Belgique, 8°, vol. 47 (1892–95). Cf. further Townsend, Göttingen Dissertation, 1900, p. 34. In the form given below for harmonic functions it was used in part by Hilbert, Über das Dirichletsche Prinzip, Festschrift, 150th anniversary of the Göttingen Academy, 1901, p. 8. Koebe refers to Ascoli, 1883; Math. Annalen, vol. 69 (1910), p. 71.

which converge for $x = a_2$. From this set is selected a set converging for $x = a_3$, etc.

We are thus let to a double array of functions

$$\begin{array}{lll} f_{1,1}(x), & f_{1,2}(x), & \cdots \\ f_{2,1}(x), & f_{2,2}(x), & \cdots \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot \end{array}$$

from which we can select in a great variety of ways a set converging in each of the points $a_1, a_2, \cdots$. For example, it would suffice to choose the functions of the principal diagonal of the array:

$$f_{1,1}(x), \quad f_{2,2}(x), \quad \cdots. \tag{3}$$

So much without the use of Condition (*b*). It is now an easy matter to show by the aid of this condition that the set (3) converges uniformly in the closed interval (a, b), and this completes the proof.

We proceed now to a second theorem, the proof of which is similar to that which has just been given, and which contains in substance the final result we desire.

Let S be an arbitrary Riemann's surface and let

$$u_1(x, y), \quad u_2(x, y), \quad \cdots$$

be a set of functions single-valued and continuous within S, and harmonic in the ordinary points of S. Furthermore, let $u_n(x, y)$, regarded as a function of the point (x, y) of S and the positive integer n, be finite:

$$|u_n(x, y)| < M,$$

where M is a constant. Then it is possible to choose from the above set of functions a set

$$u_{n_1}(x, y), \quad u_{n_2}(x, y), \quad \cdots$$

which converges at all interior points of S, and which, moreover, converges uniformly in any preassigned subregion Σ which together with its boundary lies within S.

Let (a_1, b_1) (a_2, b_2), $\cdots$ be a set of distinct ordinary interior points of S everywhere dense in S. From the set of functions $u_n(x, y)$ it is possible to choose a set which we will denote by

$$u_{1,1}(x, y), \quad u_{1,2}(x, y), \quad \cdots$$

and which converges for the point (a_1, b_1).

From this set we can now select a set

$$u_{2,1}(x, y), \quad u_{2,2}(x, y), \quad \cdots$$

which converges for the point (a_2, b_2). And so on.

We are thus led to a double array:

$$\begin{array}{lll} u_{1,1}(x, y), & u_{1,2}(x, y), & \cdots \\ u_{2,1}(x, y), & u_{2,2}(x, y), & \cdots \\ \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot \end{array}$$

The principal diagonal of this array will yield a set of functions $u_{i,i}(x, y) = u_{n_i}(x, y)$ which converge at every point (a_l, b_l) of the above set.

Let (x_0, y_0) be an ordinary interior point of S. With this point as center describe a circle C which contains in its interior and on its boundary only ordinary interior points of S, exclusive of branch points. In the interior of this circle the value of $u_n(x, y)$ for the points of the sheet of S in which C lies is given by Poisson's integral:

$$u_n(x, y) = \frac{1}{2\pi}\int_0^{2\pi} U_n \frac{(a^2 - r^2)\, d\psi}{a^2 - 2ar\cos(\theta - \psi) + r^2},$$

where U_n denotes the value of u_n on the circumferene of C, and (r, θ) are the polar coördinates of a point of C referred to (x_0, y_0) as pole.

The partial derivatives of $u_n(x, y)$ at interior points of C are given by differentiating under the sign of integration. On writing down these formulas it becomes evident *that $\partial u_n/\partial x$, $\partial u_n/\partial y$ are finite throughout a circle concentric with C and of smaller radius.*

This property of the functions $u_n(x, y)$ supplies the place of Condition (b) in the earlier theorem, and thus it is easy to prove that the set of functions $u_{n_i}(x, y)$ defined above converges uniformly throughout a suitably restricted neighborhood of the point (x_0, y_0).

The passage from this last result to the uniform convergence in the above region Σ is effected as follows. The boundary of Σ may be assumed to pass through no branch point, since otherwise Σ could be replaced by a more comprehensive region of the same nature and having this property. And now the boundary can be divided up into a finite number of arcs, each lying in a region in which the functions in question converge uniformly. Hence the functions converge uniformly along the boundary, and consequently throughout the interior, of Σ. This completes the proof.

Application to the Functions $f_n(z)$.

We are now in a position to show that from the functions $f_n(z)$ of § 3 a set

$$f_{n_1}(z), \quad f_{n_2}(z), \quad \cdots$$

can be selected converging uniformly in the arbitrary region Φ_k. In fact, we have but to write

$$f_n(z) = u_n(x, y) + i v_n(x, y), \qquad n > 2,$$

and apply the foregoing theorem to $u_n(x, y)$, Φ_2 and Φ_1 being taken as the regions S and Σ. We are thus led to a first set of functions

$$f_{1,1}(z), \quad f_{1,2}(z), \quad \cdots, \tag{4}$$

converging uniformly in Φ_1.

We now make a second application of the theorem, taking as our functions the set (4) (exclusive of $f_3(z)$ if it occurs there) and as our regions S and Σ the regions Φ_3 and Φ_2. We are thus led to a second set of functions

$$f_{2,1}(z), \quad f_{2,2}(z), \quad \cdots, \tag{5}$$

converging uniformly in Φ_2.

Repeating the process for the regions Φ_4 and Φ_3, and the functions (5) (exclusive of $f_4(z)$ if it occurs there), we obtain a third set, and so on. Thus we have a double array of functions

$$\begin{array}{lll} f_{1,1}(z), & f_{1,2}(z), & \cdots, \\ f_{2,1}(z), & f_{2,2}(z), & \cdots, \\ \cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot\quad\cdot \end{array}$$

If, now, from these we choose, say, the functions of the principal diagonal and set

$$f_{i,i}(z) = f_{n_i}(z),$$

the functions

$$f_{n_1}(z), \quad f_{n_2}(z), \quad \cdots \tag{6}$$

will form a set such as is desired. The indices n_i are independent of k, and the functions (6),—a fixed number of terms at the beginning having been suppressed if necessary,—converge uniformly in any given Φ_k.

We are thus led to a limiting function

$$\lim_{i=\infty} f_{n_i}(z) = f(z),$$

single-valued in Φ, and the next step consists in showing that by it:

$$t = f(z),$$

Φ is mapped on a single-leaved domain T of the t-plane.

§ 6. The Map Defined by $t = f(z)$.

The desired proof is furnished at once by a theorem of Hurwitz's* which says that if $\varphi(z, n)$ is analytic in a region S and continuous on the boundary, and if $\varphi(z, n)$ converges uniformly in the closed region S, the limiting function $\varphi(z)$ not vanishing at any point of the boundary, then the number of zeros of $\varphi(z)$ in S is the same as the number of zeros of $\varphi(z, n)$ in S for all values of n from a definite point on: $n \geqq m$.

Suppose, then, that $f(z)$ were to take on the same value A in two distinct points of Φ, P and Q. Let S be a region which together with its boundary lies within Φ, contains the points P and Q in its interior, and is such that $f(z)$ does not take on the value A on its boundary. Then $\varphi(z, i) = f_{n_i}(z) - A$ satisfies the conditions of Hurwitz's theorem.

As soon, however, as i is large enough for the region Φ_{n_i} to enclose S, this latter function has at most a single zero in S, since the function $t = f_{n_i}(z)$ maps Φ_{n_i} on a single-leaved region. This proves the assertion that Φ is mapped by the function $t = f(z)$ on a single-leaved region T of the t-plane.

Let

$$z = \varphi(t)$$

be the function defined by this map, i. e., the inverse of the function $t = f(z)$. Then $\varphi(t)$ is single valued in T, and the function w of § 1 goe. over into a function $w = \psi(t)$ likewise single-valued and analytic in Ts Thus the given algebraic configuration has been uniformized, and it remains merely to show that the functions $\varphi(t)$, $\psi(t)$ are automorphic, and that the boundary of T consists of a discrete set of points.

§ 7. The Function $\phi(t)$ Automorphic.

It is clear that the surface Φ admits an enumerably infinite group of conformal transformations into itself, the generators of which consist of those transformations obtained by projecting the first leaf of $\Phi_1 = F_1$ on the corresponding leaf of any duplicate of F_1 which forms a part of Φ.

To each of these transformations corresponds a transformation of T into itself which is one-to-one and conformal without exception. We proceed to prove that *each of these transformations is linear.*

It will be convenient to replace the region T by the region $\mathbf{T}$ obtained from T by the transformation

$$t = \frac{1}{x},$$

* Funktionentheorie, p. 722. Cf. also the application of this theorem in the paragraph cited.

in order that the boundary may lie in the finite region of the plane. Let

$$\varphi\left(\frac{1}{x}\right) = \omega(x); \quad \text{then} \quad z = \omega(x), \tag{1}$$

and let the transformations of $\mathbf{T}$ into itself corresponding to those above mentioned be denoted by

$$x' = \chi_n(x) \qquad n = 1, 2, \cdots. \tag{2}$$

The initial region $\Phi_1 = F_1$ of Φ is mapped by the function (1) on a region $\mathbf{T}_1$ which includes the point $x = \infty$ in its interior and is bounded by $2p$ simple closed non-intersecting analytic curves exterior to one another. Among the transformations (2) there are p which, together with their inverses, carry $\mathbf{T}_1$ over into $2p$ regions lying respectively in the $2p$ interiors of the curves which bound $\mathbf{T}_1$, each of these regions having with $\mathbf{T}_1$ a common boundary. There are further transformations (2) which carry $\mathbf{T}_1$ into regions lying respectively in the interiors of the curves which form the inner boundaries of the latter regions and having each a boundary in common with one of these regions. And so on indefinitely.

We will denote the number of the inner boundary curves at the end of the n-th step by N, the curves themselves by $C_k^{(n)}$, $k = 1, 2, \cdots, N$, and their respective lengths by $l_k^{(n)}$. Thus for the initial region $n = 1, N = 2p$. Let $\mathbf{T}_n$ be the part of $\mathbf{T}$ exterior to the curves $C_k^{(n)}$, $k = 1, 2, \cdots, N$.

The proof that the transformations (2) are linear depends essentially on the following

LEMMA. *The series*

$$\sum_{n=1}^{\infty}\sum_{k=1}^{N}[l_k^{(n)}]^2 \tag{3}$$

converges.

The proof of this lemma depends in turn on a theorem relating to the amount of deformation in certain conformal maps (*Verzerrungssatz*, Koebe), to which we now turn.

§ 8. The Amount of Deformation in Certain Conformal Maps.

Let Σ' be any single-leaved region of the z-plane not including the point $z = \infty$ in its interior, and let Σ be a region which together with its boundary lies within Σ'. Consider the functions*

$$Z = f(z)$$

* The theorem holds for a multiple-leaved region, provided Σ is taken, like the $\bar{\Sigma}$ of § 4, as a finite subregion without branch points.

which map Σ' conformally on single-leaved regions S' not including the point $Z = \infty$ in their interiors. Then, if z_1 and z_2 be any two distinct points of Σ, the ratio $f'(z_1)/f'(z_2)$ remains finite:

$$\left|\frac{f'(z_1)}{f'(z_2)}\right| < M,$$

where M is a constant.

Let O: $z = 0$, be an interior point of Σ. The function

$$F(z) = \frac{f(z) - f(0)}{f'(0)}$$

satisfies the conditions imposed on $f(z)$ in § 4, the present region Σ, or if necessary a larger one, corresponding to the region Σ of that paragraph, and it follows from that paragraph that $F'(z)$ is finite in Σ. Since

$$\frac{f'(z_1)}{f'(z_2)} = \frac{F'(z_1)}{F'(z_2)},$$

it will be sufficient to show that $|F'(z)|$ has a positive lower limit in Σ.

Suppose this were not the case. Then it would be possible to find a set of functions of the above class, $f_1(z)$, $f_2(z)$, $\cdots$, together with their related functions

$$F_n(z) = \frac{f_n(z) - f_n(0)}{f_n'(0)}, \tag{1}$$

and a set of points $z_1, z_2, \cdots$ in Σ, such that

$$|F_n'(z_n)| < \epsilon \qquad n \geqq m. \tag{2}$$

The points z_n have at least one point of condensation in Σ, and it will be convenient to take them as having but one, $\lim\limits_{n=\infty} z_n = \bar{z}$.

Let Σ_1 be a finite region lying within Σ' and enclosing Σ in its interior. Then, since $F_n(z)$ remains finite in Σ_1 by § 4, it is possible by the method of § 5 to pick out from the functions (1) a set of functions

$$F_{n_1}(z), \quad F_{n_2}(z), \quad \cdots$$

converging uniformly in Σ_1 and such that the indices n_i are independent of the particular choice of Σ_1. It follows, then, by applying Hurwitz's theorem as in § 6, that the limiting function*

$$F(z) = \lim_{i=\infty} F_{n_i}(z)$$

also corresponds to a function of the class considered in the theorem.

* That this function does not vanish identically is seen from the fact that $F_{n_i}'(0) = 1$.

We now proceed to show that

$$F'(\bar{z}) = 0.$$

From this contradiction follows the truth of the theorem.

Since $F'_{n_i}(z)$ converges uniformly in Σ, we have

$$(3) \qquad |F'_{n_i}(z_{n_i}) - F'(z_{n_i})| < \epsilon, \qquad i \geqq \mu.$$

Moreover, from the continuity of $F'(z)$ it follows that

$$(4) \qquad |F'(z_{n_i}) - F'(\bar{z})| < \epsilon, \qquad i \geqq \mu'.$$

Combining (2), written for $n = n_i$, with (3) and (4), we get

$$|F'(\bar{z})| < 3\epsilon,$$

and this completes the proof.

§ 9. Proof of the Lemma of § 7.

We are now in a position to prove the lemma of § 7. Let $l = l_k^{(1)}$, i. e., let l be the length of one of the bounding curves $C = C_k^{(1)}$ of $\mathbf{T}_1$, and let $\Sigma' = \Sigma'_1$ be a finite strip enclosing C in its interior and such that its reproductions Σ'_n under the transformations of the group (2) of § 7 do not overlap. Let $\Sigma = \Sigma_1$ be a strip lying within Σ' and enclosing C in its interior. Then it and its reproductions Σ_n lie in a finite region of the x-plane.

Let l_n be the length of the image of C when the transformation

$$x' = \chi_n(x)$$

is performed on C. Then we can write with Koebe*

$$l_n^2 = \int_0^l \int_0^l |\chi'_n(z_1)| \cdot |\chi'_n(z_2)| \cdot |dz_1| \cdot |dz_2|.$$

Applying to the integrand the algebraic relation

$$2AB \leqq A^2 + B^2,$$

we have

$$(1) \qquad l_n^2 \leqq \frac{1}{2}\int_0^l \int_0^l \left\{ |\chi'_n(z_1)|^2 + |\chi'_n(z_2)| \right\} |dz_1| \cdot |dz_2|.$$

On the other hand, consider the area f_1 of $\Sigma = \Sigma_1$ and the area f_n of Σ_n. Since the ratio of similitude in the two maps is $|\chi'_n(z)|$, we have

$$f_n = \int\int_{\Sigma} |\chi'_n(z)|^2 \, dS,$$

extended over Σ, and hence

$$f_n = |\chi'_n(\zeta)|^2 f_1,$$

* Math. Annalen, vol. 69, p. 28.

where $|\chi_n'(\zeta)|$ denotes a mean value of the integrand, such a value actually being taken on at a point $z = \zeta$ of Σ.

From the theorem of § 8 it follows that at an arbitrary point z of Σ

$$|\chi_n'(z)| < M|\chi_n'(z')|,$$

where z' denotes any second point of Σ, and hence, in particular,

$$|\chi_n'(z)|^2 < M^2 |\chi_n'(\zeta)|^2 = \frac{M^2}{f_1} f_n.$$

Applying this result to the integrand of (1), we get

$$l_n^2 \leqq \frac{M^2 l^2}{f_1} f_n.$$

But

$$\sum_{n=1}^{\infty} f_n,$$

being a series of non-overlapping areas which lie in a finite region of the plane, converges. Hence the series

$$\sum_{n=1}^{\infty} l_n^2$$

converges.

The series (3) of § 7 is the sum of the p series here obtained which correspond to the $2p$ boundaries of $\mathbf{T}_1$ taken in pairs,—a pair consisting of two boundary curves which are carried over into each other by one of the transformations (2) of § 7 and its inverse. This completes the proof of the lemma of § 7.

§ 10. Completion of the Proof that $\phi(t)$ is Automorphic.

Let

(1) $$x' = \chi(x)$$

be an arbitrary transformation of the group (2), § 7, and let

(2) $$x'' = L(x')$$

be a linear transformation that carries the point $x' = \chi(\infty)$ back to the point $x'' = \infty$. Let

(3) $$y = f(x) = L[\chi(x)].$$

Then $f(x)$ has a pole of the first order in the point $x = \infty$, and is analytic everywhere else in $\mathbf{T}$.

Let Γ be a simple closed curve that contains in its interior all the boundary points of $\mathbf{T}$. Let x be an arbitrary point of $\mathbf{T}$ lying within Γ. Choose n so that x lies within $\mathbf{T}_n$, and also that the boundary of $\mathbf{T}_n$ lies within Γ. Then $f(x)$ can be represented by Cauchy's integral formula:

$$f(x) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\xi)d\xi}{\xi - x} + \frac{1}{2\pi i}\sum_{k=1}^{N}\int_{C_k^{(n)}}\frac{f(\xi)d\xi}{\xi - x}.$$

The left hand side of this equation and the first term on the right are independent of n. Hence the second term on the right must also be independent of n. We proceed to show that it is independent of x, too, and that it has, in fact, the value 0.

Let n' be a particular value of n satisfying the above condition, and let D be the distance from x to the boundary of $\mathbf{T}_{n'}$. Then, for $n \geqq n'$ and for an arbitrary point ξ of the boundary of $\mathbf{T}_n$,

$$|\xi - x| \geqq D.$$

Let ξ' be a point of the curve $C = C_k^{(n)}$, and let l be the length of this curve. Then

$$\int_C \frac{f(\xi)d\xi}{\xi - x} = f(\xi')\int_C \frac{d\xi}{\xi - x} + \int_C \frac{f(\xi) - f(\xi')}{\xi - x}\,d\xi.$$

The first integral on the right hand side vanishes by Cauchy's integral theorem. As regards the second,

$$\left|\int_C \frac{f(\xi) - f(\xi')}{\xi - x}\,d\xi\right| \leqq \frac{1}{D}\int_C |f(\xi) - f(\xi')|\cdot|d\xi|.$$

Let Δ be the oscillation of $f(\xi)$ along C, i. e., the maximum value of $|f(z_1) - f(z_2)|$ for any two points of C. Then

$$\int_C |f(\xi) - f(\xi')|\cdot|d\xi| \leqq \Delta l \leqq \tfrac{1}{2}(\Delta^2 + l^2).$$

Hence it follows that

$$\left|\sum_{k=1}^{N}\int_{C_k^{(n)}}\frac{f(\xi)d\xi}{\xi - x}\right| \leqq \frac{1}{2D}\sum_{k=1}^{N}[(\Delta_k^{(n)})^2 + (l_k^{(n)})^2],$$

where $\Delta_k^{(n)}$ denotes the oscillation of $f(\xi)$ along $C_k^{(n)}$.

The sum on the right hand side can be made arbitrarily small, since, as will now be shown, it is the general term of a convergent series.

The series

$$\sum_{n=1}^{\infty}\sum_{k=1}^{N}[l_k^{(n)}]^2$$

converges by the lemma of § 7.

Furthermore, the curves $C_k^{(n)}$ are carried by the transformation (2) into curves of the finite plane, and thus

$$l_k'^{(n)} < G l_k^{(n)},$$

where $l_k'^{(n)}$ denotes the length of the image of $l_k^{(n)}$, and G is a constant. Hence

$$\sum_{n=1}^{\infty}\sum_{k=1}^{N}[l_k'^{(n)}]^2 \tag{4}$$

is a convergent series.

Consider now the oscillation of $f(\xi)$ along the curve $C_k^{(n)}$. By means of (1) this curve is carried into a curve $C_{k'}^{(n')}$, and the latter curve is carried by (2) into a curve C' of length $l_{k'}'^{(n')}$. Hence the oscillation of $f(\xi)$ along $C_k^{(n)}$, being equal to the maximum diameter of C', is less than half its length, or

$$\Delta_k^{(n)} < \tfrac{1}{2}l_{k'}'^{(n')}.$$

But the series

$$\sum_{n'}\sum_{k'}[l_{k'}'^{(n')}]^2$$

is made up of the same terms as the series (4). Hence the series

$$\sum_{n=1}^{\infty}\sum_{k=1}^{N}[\Delta_k^{(n)}]^2$$

converges.

We have thus arrived at the following representation of $f(x)$:

$$f(x) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\xi)d\xi}{\xi - x}.$$

Thus it appears that $f(x)$ can be continued analytically over the whole interior of Γ, and $f(x)$ is, therefore, a linear function of x. Consequently $\chi(x)$ is also a linear function of x, and the automorphic character of the function $\omega(x)$, and with it $\varphi(t)$, is herewith established.

§ 11. The Boundary of T Discrete.

Finally, it remains to show that the boundary of T is a discrete set of points.

This property follows at once for $\mathbf{T}$, and hence also for T, from the lemma of § 7, since all the boundary points of $\mathbf{T}$ are included in the interiors of the curves $C_k^{(n)}$, and the length of each of these curves can be made arbitrarily small by a suitable choice of n.

Incidentally we have obtained the result that the area of the portion of the x-plane exterior to $\mathbf{T}_n$ can be made arbitrarily small by choosing n large enough; for the area enclosed by $C_k^{(n)}$ obviously cannot exceed $[l_k^{(n)}]^2/4\pi$. Thus the boundary of $\mathbf{T}$ can be enclosed in a finite number of regions whose total area is arbitrarily small.

HARVARD UNIVERSITY,
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